The boundary integral method for the linearized rotating Navier–Stokes equations in exterior domain

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\textbf{A R T I C L E I N F O}

Keywords:
Rotating Navier–Stokes equations
Exterior domain
Boundary integral method
Finite element approximation

\textbf{A B S T R A C T}

In this paper, we apply the boundary integral method to the linearized rotating Navier–Stokes equations in exterior domain. Introducing some open ball which decomposes the exterior domain into a finite domain and an infinite domain, we obtain a coupled problem by the linearized rotating Navier–Stokes equations in finite domain and a boundary integral equation without using the artificial boundary condition. For the coupled problem, we show the existence and uniqueness of solution. Finally, we study the finite element approximation for the coupled problem and obtain the error estimate between the solution of the coupled problem and its approximation solution.

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1. Introduction

Let $\Omega^c \subset \mathbb{R}^3$ be a simply connected bounded domain with smooth boundary $\Gamma$. Assume that $\Omega^c$ is rotating with the constant angular velocity $\tilde{\omega}$ in a steady viscous incompressible fluid filling the entire space, then the steady rotating Navier–Stokes equations for the fluid occupying $\Omega = \mathbb{R}^3 \setminus \overline{\Omega^c}$ consist in finding the velocity vector $u$ and the pressure $p$ such that

\begin{equation}
\begin{aligned}
-\nu \Delta u + u \cdot \nabla u - (\tilde{\omega} \times x) \cdot \nabla u + \tilde{\omega} \times u + \nabla p &= f \\
\text{div} u &= 0 \\
u |
\end{aligned}
\end{equation}

where $\nu > 0$ is the viscous coefficient, $f$ denotes the external force acting on the fluid.

Because the problem (1) is very practical for some mathematics modelling, a large number of references study the existence, uniqueness, regularity and asymptotic behaviour in space. The interesting reader can refer to [1–8] and references therein. In these references, the problem (1) is studied from the theoretical view. However, from the numerical view, few results are developed. The reason is that the coefficient “$\tilde{\omega} \times x$” tends to infinity at large spatial distances. A natural method to overcome this difficulty is the artificial boundary method whose idea is that the exterior domain or the unbounded domain has to be truncated and the appropriate artificial boundary conditions are introduced. For example, Bonisch and Heuveline in [9–11] study the adaptive finite element method for the free fall of a solid in a fluid occupied the whole space. Dunne and Rannacher in [12,13] study the adaptive finite element method for the fluid–structure interaction. On these references, we also refer the collected paper [14]. In [14], Bonisch, Dunne and Rannacher truncate the exterior domain to a bounded domain and introduce artificial “outflow” boundary conditions governed by the Gaussian heat kernel. They analyze the effect of the diameter of the truncated computational domain on the free fall of the solid. To obtain the satisfactory

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doi:10.1016/j.amc.2010.03.112
accuracy, the diameter \( d_0 \) of the truncated domain must satisfy \( d_0 \geq 400 \) if the length of the solid is \( l = 6 \) and the width of the solid is \( w = 1 \). Hence, for the artificial boundary method, the truncated domain must be sufficiently large. A direct effect to the numerical computation is adding the CPU time. The reason that why the truncated domain is sufficiently large is that the artificial "outflow" boundary conditions are not exact on the truncated boundary. To avoid using the large truncated domain, we study the coupled method of the boundary element and the finite element, which is also called the boundary integral method.

The boundary integral method is a good method to study the numerical method for the exterior problems or the unbounded problems. A general references can be found in the work [15]. The main idea of the boundary integral method is introducing some open ball which decomposes the exterior domain into a finite domain and an infinite domain, and transforming the equation in infinite domain to the boundary integral equation without using the artificial boundary condition. There are some references about the boundary integral method for Navier–Stokes equations in exterior domain, such as [16–21] and references therein.

In this paper, we will apply the boundary integral method to the linearized rotating Navier–Stokes equations in exterior domain. Firstly, we truncate the exterior domain by an open ball, and obtain a bounded domain and a new exterior domain which is outside the ball. In the bounded domain, we consider linearized rotating Navier–Stokes equations. But in the new exterior domain, we consider the Stokes problem. Subsequently, we transform the Stokes problem into a boundary integral equation on the boundary of the ball and avoid using the artificial boundary condition. It is noteworthy that the boundary integral equation is exact on the boundary of the ball. Hence, we obtain the continuous coupled problem in terms of the boundary integral equation and the continuous coupled problem are derived in Sections 3 and 4, respectively. Moreover, the existence and uniqueness of the solution of the coupled problem and the error estimate is derived.

This paper is organized as follows. In Section 2, we describe the linearized rotating Navier–Stokes equations; the boundary integral equation and the continuous coupled problem are derived in Sections 3 and 4, respectively. Moreover, the existence and uniqueness of the solution of the coupled problem is shown in Section 4; in last section, the finite element approximation is studied for the continuous coupled problem and the error estimate is derived.

2. Linearized rotating Navier–Stokes equations

In this paper, we consider the following linearized rotating Navier–Stokes equations:

\[
\begin{aligned}
-\nabla \mathbf{u} + \omega \times \mathbf{u} - (\omega \times \mathbf{x}) \cdot \nabla \mathbf{u} + \nabla p &= f & \text{in } \Omega, \\
\text{div} \mathbf{u} &= 0 & \text{in } \Omega, \\
\mathbf{u} &= \omega \times \mathbf{x} & \text{on } \Gamma, \\
\mathbf{u}(x) &\to 0 & \text{as } x \to +\infty,
\end{aligned}
\]  

(2)

where \( f \in L^2(\Omega) \cap L^2(\Omega). \)

Without loss of generality, we assume that the origin of coordinates lies in \( \Omega^c \). Let \( \delta(\Omega^c) \) be the coordinates of \( \Omega^c \), that is

\[
\delta(\Omega^c) = \sup_{x,y \in \Omega^c} |x - y|.
\]

For each \( R > \delta(\Omega^c) \), set

\[
\Omega_R = \Omega \cap B_R, \quad \Omega^c = \mathbb{R}^3 \setminus \Omega_R,
\]

where \( B_R = \{ x \in \mathbb{R}^3, |x| \leq R \} \).

Firstly, we make the homogeneous of the boundary condition on \( \Gamma \) in (2). Silvestre in [4] showed the following lemma:

**Lemma 2.1.** For each \( R > \delta(\Omega^c) \), there exists a smooth function \( \psi \in C_0^\infty(\Omega) \) such that

\[
\begin{aligned}
di \psi &= 0 & \text{in } \Omega, \\
\psi &= \omega \times \mathbf{x} & \text{on } \Gamma, \\
\| \psi \|_q + \| \nabla \psi \|_r + \| \nabla^2 \psi \|_s &\leq \kappa_1 |\omega| & \text{for all } q \geq \frac{3}{2}, 1 < r \leq 6, 1 \leq s \leq 2,
\end{aligned}
\]

where \( \kappa_1 > 0 \) depends on \( q, r, s, \omega, \Gamma \).

Let \( u = v + \psi \) in (2), then \( v \) satisfies

\[
\begin{aligned}
-\nabla \mathbf{v} + \omega \times \mathbf{v} - (\omega \times \mathbf{x}) \cdot \nabla \mathbf{v} + \nabla p &= F & \text{in } \Omega, \\
\text{div} \mathbf{v} &= 0 & \text{in } \Omega, \\
v &= 0 & \text{on } \Gamma, \\
v(x) &\to 0 & \text{as } x \to +\infty.
\end{aligned}
\]  

(3)
where
\[ F = v \Delta \psi + (\partial \times \Psi) \cdot \nabla \psi - \partial \times \psi + f. \]

According to Lemma 2.1, we can easily show \( F \in L^2_{\text{loc}}(\Omega) \cap L^2_{\text{loc}}(\Omega) \) and
\[ \| F \| \leq \kappa_2 |\tilde{\Omega}|, \]
where \( \kappa_2 > 0 \) depends on \( v \) and \( \Gamma \).

3. The boundary integral equation

Select \( R > \delta(\Omega^e) \) such that \( \Omega_x \) contains the compact support of \( f \) and \( \psi \). The ball \( B_R \) decomposes the unbounded domain \( \Omega \) into a finite domain \( \Omega_R \) and an exterior domain \( \Omega^e \). We will use the following coupled problem to approximate the linearized rotating Navier--Stokes equation (3):
\[
\begin{align*}
- \nu \Delta v_R + \omega \times v_R - (\omega \times x) \cdot \nabla v_R + \nabla p_R &= F & \text{in } \Omega_R, \\
\text{div } v_R &= 0 & \text{in } \Omega_R, \\
- \nu \Delta p_R + \nabla \cdot D &= 0 & \text{in } \Omega^e, \\
\text{div } \nu R &= 0 & \text{in } \Omega^e, \\
v_R &= \nu R & \text{on } \Gamma, \\
\chi_R &= \chi R & \text{on } \partial B_R,
\end{align*}
\]
where
\[
\begin{align*}
\nu_R &= \nu|_{\Omega_R}, & \nu R &= \nu|_{\Omega^e}, & \nu R &= \nu|_{\partial \Omega}, \\
\chi_R &= \sum_{i=1}^{3} \sigma_i(v_R, p_R) n_i|_{\partial \Omega_R}, & \chi R &= \sum_{i=1}^{3} \sigma_i(\nu R, p R) n_i|_{\partial \Omega^e}, \\
\sigma_i(v_R, p_R) &= - \delta_0 p_R + \nu \left( \frac{\partial v_R}{\partial x_i} + \frac{\partial v_R}{\partial x_j} \right), & \sigma_i(\nu R, p R) &= - \delta_0 p R + \nu \left( \frac{\partial \nu R}{\partial x_i} + \frac{\partial \nu R}{\partial x_j} \right),
\end{align*}
\]
and \((v, p)\) is the solution of (3). Moreover, \( \nu R(x) \to 0 \) as \(|x| \to +\infty \).

Let \( (E_1(x, y), P_1(x, y)) \) \((i, j = 1, 2, 3)\) be the fundamental solutions of three dimensional Stokes problem, then they satisfy
\[
\begin{align*}
- \nu \Delta E_i(x - y) + \frac{\partial}{\partial x_i} P_i(x - y) &= \delta_0 \delta(x - y), \\
\frac{\partial}{\partial x_i} E_i(x - y) &= 0,
\end{align*}
\]
where \( \delta_0 \) is Kronecker symbol and \( \delta(x - y) \) is the Dirac function. It is well known that \((E_i(x, y), P_i(x, y))\) have the following forms
\[
\begin{align*}
E_i(x, y) &= - \frac{1}{4\pi} \frac{x_i y_j + [x_j y_i]}{|x - y|^3}, \\
P_i(x, y) &= \frac{1}{4\pi} \frac{y_i x_j}{|x - y|^3}.
\end{align*}
\]
Ladyzhenskaya in [22] showed that if \((v, p)\) is a solution of (4) in \( \Omega^e \), then for \( x \in \Omega^e \), there holds
\[
\begin{align*}
v_i(x) &= - \int_{\partial B_R} v(y) \cdot \sigma(E_i, P_i)(x - y) \cdot n(y) ds(y) + \int_{\partial B_R} E_i(x - y) \cdot \chi(y) ds(y), \quad k = 1, 2, 3, \\
p(x) &= - 2\nu \int_{\partial B_R} \frac{\partial P_i}{\partial n} (x - y) \cdot v(y) ds(y) + \int_{\partial B_R} P(x - y) \cdot \chi(y) ds(y)
\end{align*}
\]
where
\[
\begin{align*}
E_i(x - y) &= (E_{i1}(x - y), E_{i2}(x - y), E_{i3}(x - y)), \\
P(x - y) &= (P_1(x - y), P_2(x - y), P_3(x - y)), \\
\sigma_i(v, p) &= - \delta_0 p + \nu \left( \frac{\partial v}{\partial x_i} + \frac{\partial v}{\partial x_j} \right), \\
\chi &= \sum_{i=1}^{3} \sigma_i(v, p) n_i|_{\partial \Omega_R}.
\end{align*}
\]
Moreover, if \( x \in \partial B_R \), there holds
\[
\frac{1}{2} P_k(x) = - \int_{\partial B_R} v(y) \cdot \sigma(E_k, P_k)(x - y) \cdot n(y) ds(y) + \int_{\partial B_R} E_k(x - y) \cdot \chi(y) ds(y), \quad k = 1, 2, 3,
\]
which is the boundary integral equation.
4. The coupled problem

If \((v, p)\) is the solution of (4) in \(\Omega_k\), then
\[
\begin{cases}
-\nabla \Delta v + \hat{\omega} \times v - (\hat{\omega} \times x) \cdot \nabla v + \nabla p = F & \text{in } \Omega_k, \\
\text{div } v = 0 & \text{in } \Omega_k, \\
v = 0 & \text{on } \Gamma.
\end{cases}
\]
(6)

Introduce the Hilbert space
\[ V = \left\{ v \in H^1(\Omega_k)^3, \ v = 0 \text{ on } \Gamma \right\}, \]
with norm
\[ \| \nabla v \| = \left( \int_{\Omega_k} |\nabla v|^2 \, dx \right)^{1/2}. \]

Denote by \(V_0\) the solenoidal subspace of \(V\) and
\[ M = \left\{ q \in L^2(\Omega_k), \ \int_{\Omega_k} q \, dx = 0 \right\} = L_0^2(\Omega_k), \]
with \(L^2(\Omega_k)\)-norm. Introduce the following bilinear forms and trilinear form:
\[
\begin{align*}
& a(u, v) = 2\nu \sum_{ij=1}^3 \int_{\Omega_k} \varepsilon_{ij}(u) : \varepsilon_{ij}(v) \, dx, \\
& b(u, v, w) = \frac{3}{2} \sum_{i=1}^3 \int_{\Omega_k} \mathbf{w} \cdot \nabla_F \mathbf{w} \, dx, \\
& d(v, p) = \int_{\Omega_k} \mathbf{w} \cdot \nabla_F \mathbf{w} \, dx, \\
& (f, v) = \int_{\Omega_k} f v \, dx, \\
& < v, \lambda > = \sum_{i=1}^3 \int_{\partial \Omega_k} \lambda \delta_i v \, ds(x),
\end{align*}
\]
where \(< \cdot, \cdot >\) denotes the duality pairing between \(H^{-\frac{1}{2}}(\partial B_k)^3\) and \(H^{1/2}(\partial B_k)^3\), \(\varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)\). Then the variational formulation of (6) is
\[
\begin{cases}
a(v, w) - b(\hat{\omega} \times x, v, w) + (\hat{\omega} \times v, w) - d(w, p) - < w, \lambda > = (F, w) & \forall \ w \in V, \\
d(v, q) = 0 & \forall \ q \in M.
\end{cases}
\]
(7)

It is well known that (7) is equivalent to
\[
\begin{cases}
a(v, w) - b(\hat{\omega} \times x, v, w) + (\hat{\omega} \times v, w) - d(w, p) - < w, \lambda > = (F, w) & \forall \ w \in V_0.
\end{cases}
\]
(8)

On the other hand, if \((v, p)\) is the solution of (4) in \(\Omega^R\), then
\[
\begin{cases}
-\nabla \Delta v + \nabla p = 0 & \text{in } \Omega^R, \\
\text{div } v = 0 & \text{in } \Omega^R, \\
v(x) \to 0 & \text{as } x \to +\infty.
\end{cases}
\]

Set
\[ T = \left\{ \mu \in H^{-\frac{1}{2}}(\partial B_k)^3, \ \int_{\partial B_k} \mu ds = 0 \right\}. \]

Then
\[ \lambda = \left( \sum_{i=1}^3 \sigma_{ij}(v, p) n_i |_{\partial B_k} \right)_i \in T, \quad i = 1, 2, 3. \]

In fact,
\[
\int_{\partial B_k} \lambda ds = \int_{\partial B_k} \sigma(v, p) \cdot n ds = \int_{\Omega^R} \text{div } \sigma(v, p) \, dx = \int_{\Omega^R} (\nu \Delta v - \nabla p) \, dx = 0.
\]
Eq. (5) multiplied by \( \mu \in T \) and integrated over \( \partial B_k \) yields
\[
2b(\lambda, \mu) - < v, \mu > - 2 < Gv, \mu > = 0 \quad \forall \mu \in T,
\]
where \( G = \{G_1, G_2, G_3\} \), and
\[
\begin{align*}
  b(\lambda, \mu) &= \sum_{i=1}^{3} \int_{\partial B_k} \int_{\partial B_k} \mu_k(x) E_\lambda(x-y) \lambda_i(y) ds(y) ds(x), \\
  G_k(v) &= \int_{\partial B_k} v(y) \cdot \sigma(E_k, P) (x-y) \cdot n(y) ds(y).
\end{align*}
\]

Then the variational formulation of the coupled problem by the linearized rotating Navier–Stokes equation (6) in finite domain \( \Omega_k \) and the boundary integral equation (5) is
\[
\begin{align*}
  \text{Find } (v, \lambda, p) \in V \times T \times M \text{ such that } \\
  a(v, w) - b(\bar{\omega} \times x, v, w) + (\bar{\omega} \times v, w) + d(w, p) + < w, \lambda > &= (F, w) \quad \forall w \in V, \\
  2b(\lambda, \mu) - < v, \mu > - 2 < Gv, \mu > &= 0 \quad \forall \mu \in T, \\
  d(v, q) &= 0 \quad \forall q \in M.
\end{align*}
\]
Moreover, the variational formulation (10) is equivalent to
\[
\begin{align*}
  \text{Find } (v, \lambda) \in V_\sigma \times T \text{ such that } \\
  a(v, w) - b(\bar{\omega} \times x, v, w) + (\bar{\omega} \times v, w) + < w, \lambda > &= (F, w) \quad \forall w \in V_\sigma, \\
  2b(\lambda, \mu) - < v, \mu > - 2 < Gv, \mu > &= 0 \quad \forall \mu \in T.
\end{align*}
\]
The following lemma due to Li and He in [16] plays a key role in establishing the existence of solution to (11).

**Lemma 4.1.** The bilinear form \( b(\cdot, \cdot) \) is symmetric and continuous on \( T \times T \). Moreover, there exists a positive constant \( c > 0 \) such that
\[
b(\mu, \mu) \geq c\|\mu\|_{\mathcal{H}(\partial B_k)}^2 \quad \forall \mu \in T.
\]

Then for given \( \nu \in V_\sigma \), the problem
\[
\begin{align*}
  \text{Find } \lambda \in T \text{ such that } \\
  2b(\lambda, \mu) - < v, \mu > - 2 < Gv, \mu > &= 0 \quad \forall \mu \in T,
\end{align*}
\]
admits a unique solution \( \lambda = \iota(\nu) \). Hence, the problem (11) is equivalent to
\[
\begin{align*}
  \text{Find } v \in V_\sigma \text{ such that } \\
  a(v, w) - b(\bar{\omega} \times x, v, w) + (\bar{\omega} \times v, w) + < w, \iota(\nu) > &= (F, w) \quad \forall w \in V_\sigma.
\end{align*}
\]
On \( \iota(\nu) \), we have

**Lemma 4.2.** For given \( \nu \in V_\sigma \), there holds
\[
< v, \iota(\nu) > \geq 0.
\]

**Proof.** Consider the following problem
\[
\begin{align*}
- \nu \Delta w + \nabla \pi &= 0 \quad \text{in } \Omega_k, \\
\text{div} w &= 0 \quad \text{in } \Omega_k, \\
w|_{\Gamma} &= 0, \quad w|_{\partial B_k} = v|_{\partial B_k}.
\end{align*}
\]
It is easy to show that the problem (14) exists a unique solution \( w \in V_\sigma \). Then
\[
< v, \iota(\nu) > = \int_{\partial B_k} \sigma(\nu, p)n \nu ds = \int_{\partial B_k} \sigma(\nu, \pi)n \nu ds = \int_{\partial B_k} \text{div} (\sigma(\nu, \pi) \cdot w) dx = \int_{\partial B_k} \text{div} \sigma(\nu, \pi) \cdot w dx + \int_{\partial B_k} \sigma(\nu, \pi) \epsilon(w) dx = \int_{\partial B_k} \sigma(\nu, \pi) \epsilon(w) dx = 2v \int_{\partial B_k} \epsilon(\nu) \epsilon(w) dx \geq 0. \quad \Box
\]

In terms of Lemma 4.2, setting \( w = v \) in (13) yields
\[
a(v, v) - b(\bar{\omega} \times x, v, v) + (\bar{\omega} \times v, v) + < v, \iota(\nu) > \geq v \| v \|^2_\nu.
\]
here we use \( (\bar{\omega} \times x, v, v) = 0 \) and
Lemma 5.1. The discrete problem \((10)\) is equivalent to \((17)\). Then according to Theorem 1.4.3 in [23], we can show

\[
\|v\|_V \leq \frac{K_2}{V} |\partial|.
\]

5. Finite element approximation

Denote by \(V_h, M_h\) and \(T_h\) three finite element subspaces which satisfy

\[
V_h \subset V, \quad M_h \subset M, \quad T_h \subset T,
\]

and discrete inf-sup condition, that is, there exists a positive constant \(\beta > 0\), independent of \(h\), such that

\[
\beta \|q_h\|_{0,\Omega_h} \leq \sup_{w_h \in V_h} \frac{d(w_h, q_h)}{|w_h|_V} \quad \forall q_h \in M_h,
\]

where \(\|\cdot\|_{m,\Omega} \) is the \(H^m(\Omega_h)\)-norm.

The finite element discrete formulation of \((10)\) is

\[
\begin{aligned}
&\text{Find } (v_h, \lambda_h, p_h) \in V_h \times T_h \times M_h \text{ such that } \\
& a(v_h, w_h) - b(\partial \alpha \times x, v_h, w_h) + (\partial \alpha \times v_h, w_h) + d(w_h, p_h) + \langle w_h, \lambda_h \rangle = (F, w_h) \quad \forall w_h \in V_h, \\
& 2b(\lambda_h, \mu_h) - \langle \mu_h, \mu_h \rangle > -2 < G v_h, \mu_h > 0 \quad \forall \mu_h \in T_h, \\
& d(v_h, q_h) = 0 \quad \forall q_h \in M_h.
\end{aligned}
\]

Then according to Theorem 1.4.3 in [23], we can show

**Theorem 4.1.** The problem \((11)\) exists a unique solution \((\nu, \lambda) \in V \times T\). Moreover, \(\nu\) satisfies

\[
\|v\|_V \leq \frac{K_2}{V} |\partial|.
\]

Set \(V_{oh} = \{w_h \in V_h, \quad d(w_h, q_h) = 0 \quad \forall q_h \in M_h\}\). Then \((15)\) is equivalent to

\[
\begin{aligned}
&\text{Find } (v_h, \lambda_h) \in V_{oh} \times T_h \text{ such that } \\
& a(v_h, w_h) - b(\partial \alpha \times x, v_h, w_h) + (\partial \alpha \times v_h, w_h) + \langle w_h, \lambda_h \rangle = (F, w_h) \quad \forall w_h \in V_{oh}, \\
& 2b(\lambda_h, \mu_h) - \langle \mu_h, \mu_h \rangle > -2 < G v_h, \mu_h > 0 \quad \forall \mu_h \in T_h.
\end{aligned}
\]

Because \(V_{oh}\) is not the subspace of \(V\), we show that for \(v_h \in V_{oh}\), Lemma 4.2 also holds.

**Lemma 5.1.** For given \(v_h \in V_{oh}\), we have

\[
<v_h, \lambda(v_h) > \geq 0.
\]

**Proof.** Consider the following

\[
\begin{aligned}
&\text{Find } (w, \pi) \in V_h \times M_h \text{ such that } \\
& -\nu \Delta w + \nabla \pi = 0 \quad \text{in } \Omega_h, \\
& d(w, q) = 0 \quad \forall q \in M_h, \\
& w|_{\Gamma} = 0, \quad w|_{\partial \Omega_h} = v_h|_{\partial \Omega_h}.
\end{aligned}
\]

It is easy to show that \((17)\) exists a unique solution \((w, \pi) \in V_h \times M_h\). Similarly with the proof of Lemma 4.2, we can show

**Lemma 5.1.**

Then for discrete problem \((15)\), we have:

**Theorem 5.1.** The discrete problem \((15)\) exists a unique solution \((v_h, \lambda_h, p_h) \in V_h \times T_h \times M_h\), where \(\lambda_h = \lambda(v_h)\).

Next, we give the finite element error estimates. Suppose the following basic hypothesis hold:

(H1) There exists an interpolation operator \(R_h : V \rightarrow V_h\) such that

\[
d(q, \nu) - R_h \nu = 0 \quad \forall q \in M_h, \\
\|\nu - R_h \nu\|_V \leq c\|\nu\|_{2,\Omega_h} \quad \forall \nu \in H^2(\Omega_h) \cap V,
\]

where \(c > 0\) is independent of \(h\).

(H2) There exists an interpolation operator \(S_h : L^2(\partial \Omega_h) \rightarrow \Omega_h\) such that

\[
\|\mu - S_h \mu\|_{1,\partial \Omega_h} \leq c\|\mu\|_{1,\partial \Omega_h} \quad \forall \mu \in H^2(\partial \Omega_h) \cap \Omega_h.
\]
where \( c > 0 \) is independent of \( h \).

\textbf{(H3)} There exists an interpolation operator \( \Pi_h : M \rightarrow M_h \) such that

\[
\| q - \Pi_h q \|_{0,h_k} \leq c \| q \|_{l,h_k} \quad \forall \ q \in H^1(\Omega_k) \cap M,
\]

where \( c > 0 \) is independent of \( h \).

\textbf{Theorem 5.2.} Suppose \textbf{(H1)}--\textbf{(H3)} hold. If \((v, \lambda, p) \in H^2(\Omega_k)^2 \cap V \times H^1(\Omega_k)\cap T \times H^1(\Omega_k) \cap M\) and \((v_h, \lambda_h, p_h) \in V_h \times T_h \times M_h\) are the solutions of \((10)\) and \((15)\), respectively, then we have

\[
\begin{align*}
\| v - v_h \|_V + \| \lambda - \lambda_h \|_{1/4,h} & \leq c(1 + R^2)\delta, \\
\| p - p_h \|_{0,h} & \leq c(1 + R^4)\delta,
\end{align*}
\]

where \( c > 0 \) is independent of \( h \).

\textbf{Proof.} According to \((10)\) and \((15)\), for all \( w_h \in V_h, \mu_h \in T_h \) and \( q_h \in M_h \), we have

\[
\begin{align*}
a(v - v_h, w_h) + b(\vec{\omega} \times x, v - v_h, w_h) - d(w_h, p - p_h) + \langle w_h, \lambda - \lambda_h \rangle & = 0, \\
2b(\lambda - \lambda_h, \mu_h) + \langle v - v_h, \mu_h \rangle - 2\langle G(v - v_h), \mu_h \rangle & = 0, \\
d(v - v_h, q_h) & = 0.
\end{align*}
\]

Inserting the operators \( R_h, S_h \) and \( \Pi_h \) in \((18)\) gives

\[
a(R_h v - v_h, w_h) - b(\vec{\omega} \times x, R_h v - v_h, w_h) - d(w_h, \Pi_h p - p_h) + \langle w_h, \lambda(R_h v - v_h) \rangle \\
= a(R_h v - v_h, w_h) - b(\vec{\omega} \times x, R_h v - v_h, w_h) - d(w_h, \Pi_h p - p_h) + \langle w_h, \lambda(R_h v - v_h) \rangle,
\]

and

\[
2b(S_h \lambda - \lambda_h, \mu_h) = \langle v - v_h, \mu_h \rangle + 2\langle G(v - v_h), \mu_h \rangle + 2b(S_h \lambda - \lambda_h, \mu_h),
\]

and

\[
d(R_h v - v_h, q_h) = 0.
\]

According to \textbf{Lemma 4.1}, taking \( \mu_h = S_h \lambda - \lambda_h \) in \((20)\) yields

\[
\| S_h \lambda - \lambda_h \|_{1/4,h} \leq c \left( \| v - v_h \|_V + \| S_h \lambda - \lambda_h \|_{1/4,h} \right),
\]

where \( c > 0 \) is independent of \( h \). Hence,

\[
\| \lambda - \lambda_h \|_{1/4,h} \leq c \left( \| v - v_h \|_V + \| S_h \lambda - \lambda_h \|_{1/4,h} \right).
\]

From \textbf{Lemma 5.1}, taking \( w_h = R_h v - v_h \) in \((19)\) and \( q_h = \Pi_h p - p_h \) in \((21)\) gives

\[
\| v \|_{R_h v - v_h} \leq a(R_h v - v_h, v - v_h) - b(\vec{\omega} \times x, R_h v - v_h, R_h v - v_h) + \langle \vec{\omega} \times (R_h v - v_h), R_h v - v_h \rangle \\
- d(R_h v - v_h, \Pi_h p - p_h) + \langle R_h v - v_h, \lambda(R_h v - v_h) \rangle.
\]

Next, we will estimate the right-hand side of above inequality.

\[
a(R_h v - v, R_h v - v_h) \leq \| R_h v - v \|_V \| R_h v - v_h \|_V \leq \delta \| R_h v - v_h \|_V^2 + \frac{\delta^2}{4} \| R_h v - v \|_V^2,
\]

where \( \delta > 0 \) will be determined later.

\[
b(\vec{\omega} \times x, R_h v - v, R_h v - v_h) \leq \| \vec{\omega} \times x \|_{l,\Omega_k} \| R_h v - v \|_V \| R_h v - v_h \|_V \leq c R^2 \| R_h v - v \|_V \| R_h v - v_h \|_V \\
\leq \delta \| R_h v - v_h \|_V^2 + \frac{c R^4}{\delta} \| R_h v - v \|_V^2,
\]

where \( c > 0 \) is independent of \( h \) and \( R \).

\[
(\vec{\omega} \times (R_h v - v), R_h v - v_h) \leq c \| R_h v - v \|_V \| R_h v - v_h \|_V \leq \delta \| R_h v - v_h \|_V^2 + \frac{c}{\delta} \| R_h v - v \|_V^2,
\]

where \( c > 0 \) is independent of \( h \) and \( R \).

\[
d(R_h v - v_h, \Pi_h p - p) \leq c \| R_h v - v_h \|_V \| \Pi_h p - p \|_{0,h_k} \leq \delta \| R_h v - v_h \|_V^2 + \frac{c}{\delta} \| \Pi_h p - p \|_{0,h_k}^2,
\]

where \( c > 0 \) is independent of \( h \) and \( R \).
\[ \langle R_h v - v_h, \lambda (R_h v - v) \rangle \leq \| R_h v - v_h \|_{L^2(\partial \Omega)} \| \lambda (R_h v - v) \|_{L^2(\partial \Omega)} \leq c \| R_h v - v_h \|_{L^2(\partial \Omega)} \| R_h v - v \|_{L^2(\partial \Omega)} \]
\[ \leq c \| R_h v - v_h \|_V \| R_h v - v \|_V \leq \delta \| R_h v - v_h \|_V^2 + \frac{c}{\delta} \| R_h v - v \|_V^2, \]

where \( c > 0 \) is independent of \( h \) and \( R \). According to above estimates, for sufficiently small \( \delta > 0 \), we have
\[ \| R_h v - v_h \|^2_V \leq c (1 + R^2) \| R_h v - v \|_V^2 + c \| I_h p - p \|_{0,\partial \Omega}^2, \]

where \( c > 0 \) is independent of \( h \) and \( R \). Hence,
\[ \| v - v_h \|_V \leq c (1 + R^2) \| R_h v - v \|_V + c \| I_h p - p \|_{0,\partial \Omega}. \]

From the hypothesis (H1)–(H3) and (22), we obtain
\[ \| v - v_h \|_V + \| \lambda - \lambda_h \|_{L^2(\partial \Omega)} \leq c (1 + R^2) h. \]

where \( c > 0 \) is independent of \( h \) and \( R \). According to (19) and discrete inf-sup condition, we have
\[ \| I_h p - p_h \|_{0,\partial \Omega} \leq c (1 + R^2) (\| R_h v - v_h \|_V + \| R_h v - v \|_V) + c \| I_h p - p_h \|_{0,\partial \Omega} + \| I_h p - p \|_{0,\partial \Omega}. \]

Then, from the hypothesis (H1)–(H3), we have
\[ \| p - p_h \|_{0,\partial \Omega} \leq c (1 + R^2) (\| R_h v - v_h \|_V + \| R_h v - v \|_V) + c \| I_h p - p_h \|_{0,\partial \Omega} + \| I_h p - p \|_{0,\partial \Omega} \]
\[ \leq c (1 + R^2) \| v - v_h \|_V + \| R_h v - v \|_V + c \| I_h p - p_h \|_{0,\partial \Omega} + \| I_h p - p \|_{0,\partial \Omega} \leq c (1 + R^2) h \leq c (1 + R^4) h, \]

where \( c > 0 \) is independent of \( h \) and \( R \). \( \square \)

Acknowledgement

Supported by National Natural Science Foundation of China (Nos. 10901122 and 10971165).

References